

Statistics at Square One

XIV—The χ^2 tests

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The distribution of a discrete variable in a sample often needs to be compared with the distribution of a discrete variable in another sample.

For example, over a period of two years Dr Gold has classified by socio-economic class the women aged 20-64 admitted to his unit suffering from self-poisoning—sample A. At the same time he has likewise classified the women of similar age admitted to a gastroenterological unit in the same hospital—sample B. He has employed the Registrar General's five socio-economic classes, and generally classified the woman by reference to her father's or husband's occupation. The results are set out in table 14.1.

The problem Dr Gold wants to investigate is whether the distribution of the patients by social class differed in these two units. He therefore erects the null hypothesis that there is no difference between the two distributions. This is what he tests by chi-square (χ^2).

It is important to emphasise here that χ^2 tests may be carried out for this purpose only on the *actual numbers* of occurrences, *not* on percentages, proportions, means of observations, or other derived statistics. (There are some quite different purposes for which the χ^2 distribution is used, but they do not concern us here.)

The χ^2 test is carried out in the following steps:

For each observed number (O) in the table find an "expected" number (E); this procedure is discussed below.

Subtract each expected number from each

observed number O - E
 Square the difference (O - E)²

Divide the squares so obtained for each cell of the table by the expected number for that cell . . . (O - E)²/E
 χ^2 is the sum of (O - E)²/E.

To calculate the expected number for each cell of the table consider the null hypothesis. In this case it is that the numbers in each cell are proportionately the same in sample A as in sample B. We therefore construct a parallel table in which the proportions are exactly the same for sample A as for sample B. This is done in columns (2) and (3) of table 14.2. The proportions are obtained from the totals column in table 14.1 and they are then applied to the totals row. For instance, in table 14.2, col (2), $11.80 = (22 \div 289) \times 155$; $24.67 = (46 \div 289) \times 155$; and so on. Likewise in column (3) $10.20 = (22 \div 289) \times 134$; $21.33 = (46 \div 289) \times 134$; and so on.

Thus by simple proportions from the totals we find an expected number to match each observed number. The sum of the expected numbers for each sample must equal the sum of the observed numbers for each sample, which is a useful check. We now subtract each expected number from its corresponding observed number. The results are given in columns (4) and (5)

TABLE 14.1—Distribution by socio-economic class of patients admitted to self-poisoning (sample A) and gastroenterological (sample B) units

Socio-economic class	Samples		Total
	A	B	
I	17	5	22
II	25	21	46
III	39	34	73
IV	42	49	91
V	32	25	57
Total	155	134	289

of table 14.2. Here two points may be noted. The sum of these differences always equals 0 in each column, and each difference for sample A is matched by the same figure, but with opposite sign, for sample B. Again these are useful checks.

Then the figures in columns (4) and (5) are each squared and divided by the corresponding expected numbers in columns (2) and (3). The results are given in columns (6) and (7) of table 14.2. Finally these results, (O-E)²/E, are added. The sum of them is χ^2 .

TABLE 14.2—Calculation of chi-square on figures in table 14.1

Class (1)	Expected numbers		O-E		(O-E) ² /E	
	A (2)	B (3)	A (4)	B (5)	A (6)	B (7)
I	11.80	10.20	5.20	-5.20	2.292	2.651
II	24.67	21.33	0.33	-0.33	0.004	0.005
III	39.15	33.85	-0.15	0.15	0.001	0.001
IV	48.81	42.19	-6.81	6.81	0.950	1.009
V	30.57	26.43	1.43	-1.43	0.067	0.077
Total	155.00	134.00	0	0	3.314	3.833

$\chi^2 = 3.314 + 3.833 = 7.147$. DF=4. $0.50 > P > 0.10$.

A helpful technical procedure in calculating the expected numbers may be noted here. Most electronic calculators allow successive multiplication by a constant multiplier to be carried out by a short-cut of some kind. To calculate the expected numbers a constant multiplier for each sample is obtained by dividing the total of the sample by the grand total for both the samples. In table 14.1 for sample A this is $155 \div 289 = 0.5363$ This fraction is then successively multiplied by 22, 46, 73, 91, and 57. For sample B the fraction is $134 \div 289 = 0.4636$ This too is successively multiplied by 22, 46, 73, 91, and 57. The results are in table 14.2, columns (2) and (3).

Having obtained a value for $\chi^2 = \sum \{ (O-E)^2/E \}$ we look up in a table of χ^2 distribution the probability attached to it. Just as with the *t* table, we must enter the χ^2 table at a certain number of degrees of freedom. To ascertain these requires some care.

TABLE 14.3—Distribution of χ^2

DF	Probability					
	0.50	0.10	0.05	0.02	0.01	0.001
1	0.455	2.706	3.841	5.412	6.635	10.827
2	1.386	4.605	5.991	7.824	9.210	13.815
3	2.366	6.251	7.815	9.837	11.345	16.268
4	3.357	7.779	9.488	11.668	13.277	18.465
5	4.351	9.236	11.070	13.388	15.086	20.517
6	5.348	10.645	12.592	15.033	16.812	22.457
7	6.346	12.017	14.067	16.622	18.475	24.322
8	7.344	13.362	15.507	18.168	20.090	26.125
9	8.343	14.684	16.919	19.679	21.666	27.877
10	9.342	15.987	18.307	21.161	23.209	29.588
11	10.341	17.275	19.675	22.618	24.725	31.264
12	11.340	18.549	21.026	24.054	26.217	32.909
13	12.340	19.812	22.362	25.472	27.688	34.528
14	13.339	21.064	23.685	26.873	29.141	36.123
15	14.339	22.307	24.996	28.259	30.578	37.697
16	15.338	23.542	26.296	29.633	32.000	39.252
17	16.338	24.769	27.587	30.995	33.409	40.790
18	17.338	25.989	28.869	32.346	34.805	42.312
19	18.338	27.204	30.144	33.687	36.191	43.820
20	19.337	28.412	31.410	35.020	37.566	45.315
21	20.337	29.615	32.671	36.343	38.932	46.797
22	21.337	30.813	33.924	37.659	40.289	48.268
23	22.337	32.007	35.172	38.968	41.638	49.728
24	23.337	33.196	36.415	40.270	42.980	51.179
25	24.337	34.382	37.652	41.566	44.314	52.620
26	25.336	35.563	38.885	42.856	45.642	54.052
27	26.336	36.741	40.113	44.140	46.963	55.476
28	27.336	37.916	41.337	45.419	48.278	56.893
29	28.336	39.087	42.557	46.693	49.588	58.302
30	29.336	40.256	43.773	47.962	50.892	59.703

Table 14.3 is taken by permission of the authors and publishers from table IV of Fisher and Yates, *Statistical Tables for Biological, Agricultural and Medical Research*, published by Longman Group Ltd, London (previously published by Oliver & Boyd, Edinburgh).

When a comparison is made between one sample and another, as in table 14.1, a simple rule for the degrees of freedom is that they equal (number of columns minus 1) × (number of rows minus 1). For Dr Gold's data in table 14.1 this rule gives (2 - 1) × (5 - 1) = 4. Another way of looking at this is to ask what is the minimum number of figures that must be supplied in table 14.1, in addition to all the totals, to allow us to complete the whole table. Four numbers disposed anyhow in samples A and B provided they are in separate rows will suffice.

An abbreviated version of the χ^2 table is given in table 14.3. Entering this at 4 degrees of freedom and reading along the row we find that Dr Gold's value of χ^2 , 7.147, lies between 3.357 and 7.779. The corresponding probability is: 0.50 > P > 0.10. This is well above the conventionally significant level of 0.05, or 5%. So the null hypothesis is *not* disproved. It is therefore quite conceivable that in the distribution of the patients between socio-economic classes the population from which sample A was drawn did not differ significantly from the population from which sample B was drawn.

Quick method

The above method of calculating χ^2 illustrates the nature of the statistic clearly and is often used in practice. But a quicker method devised by Snedecor and Irwin¹ is particularly suitable for use with electronic calculators.

The data are set out as in table 14.1. Take the left-hand column of figures (sample A) and call each observation a. Their total, which is 155, is then Σa .

Let p = the proportion formed when each observation a is divided by the corresponding figure in the Total column. Thus here p in turn equals 17/22, 25/46. . . . 32/57.

Let \bar{p} = the proportion formed when the total of the observations in the left-hand column, Σa , is divided by the total of all the observations. Here $\bar{p} = 155/289$. Let $\bar{q} = 1 - \bar{p}$, which is the same as 134/289. Then

$$\chi^2 = \frac{\Sigma pa - \bar{p}\Sigma a}{\bar{p}\bar{q}}$$

The procedure with table 14.1 is as follows:

Calculate $\frac{17^2}{22}$ and store in memory
 " $\frac{25^2}{46}$ " " " "
 " $\frac{39^2}{73}$ " " " "
 " $\frac{42^2}{91}$ " " " "
 " $\frac{32^2}{57}$ " " " "
 " $\frac{155^2}{289}$ and subtract from memory.

Withdraw result from memory on to display screen = 1.776975.

We now have to divide this by $\bar{p} \times \bar{q}$. Here $\bar{p} = \frac{155}{289}$ and $\bar{q} = \frac{134}{289}$. So instead of dividing by these fractions we turn them upside down and multiply by them. Thus without removing 1.776975 from the display screen we carry out the following:

$$1.776975 \times \frac{289}{155} \times \frac{289}{134}$$

This gives us $\chi^2 = 7.146$.

The calculation naturally gives the same result if the figures for sample B are used instead of those for sample A.

Exercise 14. In a trial of a new drug against a standard drug for the treatment of depression the new drug caused some improvement in 56% of 73 patients and the standard drug some improvement in 41% of 70 patients. The results were assessed in five categories as follows: New treatment: much improved 18, improved 23, unchanged 15, worse 9, much worse 8; standard treatment: much improved 12, improved 17, unchanged 19, worse 13, much worse 9. What is χ^2 on this distribution; how many degrees of freedom are there; what is the value of P? *Answer:* 3.295; 4; P > 0.5.

Reference

¹ Snedecor, G W, and Irwin, M R, *Iowa State College of Science*, 1933, 8, 75.

What is Takayashu's syndrome?

Takayashu's syndrome is a chronic non-specific obliterative arteritis which classically affects the branches of the aortic arch at their origins; hence the synonym "aortic arch syndrome." It is also well known as pulseless disease because all the arm pulses as well as the carotids may be impalpable. A Japanese ophthalmologist, Takayashu, described it in young women. Neurological symptoms are common and may include transient failure of vision associated with the low pressure in the retinal arteries. Since the original description of a disorder affecting the cranial and upper limb arteries, arteriographic studies have shown that there may be multiple stenoses involving the aorta itself, the renal arteries, or the iliac arteries. The same or indistinguishable disorder has been described from Malaysia as the middle aortic syndrome, and the condition is often called "Oriental arteritis" to cover all types of arterial sites, ages, and sexes affected. It is not confined to young women but it is not arteriosclerotic and is seen more often in the young than in the elderly. It is active for a period and then burns out leaving the residual ischaemic legacies which may cause hypertension, a Leriche syndrome, or hemiplegia. It is fairly common everywhere in the Middle and Far East and occurs in South Africa among the Coloured and Black populations. It is very rare in Caucasians but a similar disorder is occasionally seen. The same clinical syndrome has been encountered several times in White patients with systemic lupus. The origin is unknown, there is no specific treatment, but surgical treatment to relieve ischaemia may be helpful in some cases.